New Relations Using Modified Bessel Functions with Application to pn Junction Circuits

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Abstract:

A method for generating relations between modified Bessel functions has been revealed [2]. These identities are generated on a solid physical basis. Circuits comprising pn junctions exhibit intermodulation products in the form of modified Bessel functions. A spectrum line can be described as a base harmonic or as it is generated from other spectrum lines as a consequence of nonlinear and/or time variant components in the circuit. We neglect the higher order products and keep only that of the lowest possible order and make it equal to the mentioned base harmonic. For that reason, our equations are of approximate nature, but they are quite useful in the analysis of pn junction circuits as our examples show it.

We obtain relations for frequency doubling, tripling and third order intermodulation and verify them using the harmonic balance analysis of a one diode circuit.

1. Introduction:

Recently we published a paper about the phase of the third order intermodulation [2]. Here we generalize the method used in that paper. Our newly developed relations using modified Bessel functions will exceed the content of the best available handbook on that topic [1].

State of the art: Best information source about properties of Bessel functions is [1]. Application of Bessel functions for pn junction circuits is well known [4, 5, 6]. A problem was that phase of intermodulation products has not been modelled yet, and we solved this problem [2]. Compared to the known results, in this paper such relations have been revealed containing Bessel functions, that can predict any intermodulation products at the price of computer solution of a single nonlinear equation.

We assume the voltage on the diode in the following form:

\[ uD = A_0 + A_1 \cos(\omega_1 t - \varphi_1) + A_2 \cos(\omega_2 t - \varphi_2) + A_3 \cos(\omega_3 t - \varphi_3) \]  

We assume an ideal diode:

\[ iD = I_0 \left( e^{uD/kT} - 1 \right) \]
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where \( I_0 \) and \( nVT=nnT/q \) are the reverse saturation current and the modified thermal voltage, respectively. \( k \), \( T \) and \( q \) are the Boltzmann constant, absolute temperature in \( K^0 \) and the electron charge, respectively. Temporarily we assume that \( A_2=A_3=0 \).

\[
\begin{align*}
\omega D &= A_0 + A_1 \cos(\omega_1 t - \varphi_1) = \\
&= A_0 + A_1 \cos \varphi_1 \cos \omega_1 t + A_1 \sin \varphi_1 \sin \omega_1 t
\end{align*}
\]

\[
iD = 10 \left( e^{A_0 + A_1 \cos \omega_1 t} e^{A_1 \sin \varphi_1 \sin \omega_1 t} - 1 \right)
\]

Where all \( A_i \)'s are normalized to \( nVT \).

Fourier series of an exponential nonlinearity [1] (9.6.34-35):

\[
e^\cos \omega t = I_0(V) + 2 \sum_{k=1}^\infty I_k(V) \cos k\omega t
\]

\[
e^\sin \omega t = I_0(V) +
\]

\[
+ 2 \sum_{k=0}^\infty (-1)^k I_{2k+1}(V) \sin(2k+1)\omega t + 2 \sum_{k=1}^\infty (-1)^k I_{2k}(V) \cos 2k\omega t
\]

The diode current is

\[
iD = 10 \left( e^{A_0 + A_1 \cos(\omega_1 t - \varphi_1) + A_2 \cos(\omega_2 t - \varphi_2) + A_3 \cos(\omega_3 t - \varphi_3) + \ldots} - 1 \right) =
\]

\[
= 10 \left[ e^{A_0} [I_0(A_1 \cos(\varphi_1)) + 2I_1(A_1 \cos(\varphi_1)) \sin \omega_1 t + \ldots] \ast
\]

\[
* \left[ I_0(A_2 \cos(\varphi_2)) + 2I_1(A_2 \cos(\varphi_2)) \sin \omega_2 t + \ldots \right] \ast
\]

\[
* \left[ I_0(A_3 \cos(\varphi_3)) + 2I_1(A_3 \cos(\varphi_3)) \sin \omega_3 t + \ldots \right] \ast
\]

\[
* \left[ I_0(A_3 \sin(\varphi_3)) + 2I_1(A_3 \sin(\varphi_3)) \sin \omega_3 t + \ldots \right] \ast \ldots - 1
\]

(7)

2. Frequency doubling:

Here \( \omega_2 = 2\omega_1 \) and \( A_3=0 \). We consider here a purely resistive circuit: \( \varphi_1 = \varphi_2 = 0 \).

Making equal all \( \varphi \) values to zero, does not mean that all \( A_i \)'s, \( i=1,2,\ldots \) are positive. A negative \( A_i \) also satisfies the requirement for a resistive circuit. This fact explains why the negative sign in Eq. (15,17,30) will not be an error.

\[
iD = 10 \left[ e^{A_0} [I_0(A_1) + 2I_1(A_1) \cos \omega_1 t + 2I_2(A_1) \cos 2\omega_1 t \ldots] \ast
\]

\[
* \left[ I_0(A_2) + 2I_1(A_2) \cos \omega_2 t + \ldots \right] \ast \ldots - 1 \right] =
\]

\[
= 10 \left[ e^{A_0} [I_0(A_1)] [I_0(A_2)] - 1 +
\]

\[
+ e^{A_0} [I_0(A_1)] 2I_1(A_2) \cos \omega_2 t +
\]

\[
+ e^{A_0} [I_0(A_2)] 2I_1(A_1) \cos \omega_1 t +
\]

\[
+ e^{A_0} [I_0(A_2)] 2I_2(A_1) \cos 2\omega_1 t + \ldots \right]
\]
We intend to prove analytically the following analysis result:

$$V_g \cos \omega_1 t = Z_0 \cdot iD + uD$$

DC component:

$$0 = Z_0 \cdot I_0 \cdot \left( e^{\lambda_0} [I_0(A_1)] [I_0(A_2)] - 1 \right) + A_0$$

$$\omega_1$$ component:

$$V_g = Z_0 \cdot I_0 \cdot e^{\lambda_0} [I_0(A_2)] [2I_1(A_1)] + A_1$$

$$\omega_2$$ component:

$$0 = Z_0 \cdot I_0 \cdot \left( e^{\lambda_0} [I_0(A_1)] [2I_1(A_2)] + e^{\lambda_0} [I_0(A_2)] [2I_2(A_1)] \right) + A_2$$
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With very small $A_2$, Eq. (10,11) can be solved separately:

\begin{align*}
0 &= Z_0 \cdot I_0 \cdot (e^{A_0}[I_0(A_1)] - 1) + A_0 \\
Vg &= Z_0 \cdot I_0 \cdot e^{A_0}2I_1(A_1) + A_1
\end{align*}

(13)

(14)

After solving Eq. (13 and 14) for $A_0$ and $A_1$, we can compute $A_2$ from Eq. (12).

Results are in Fig. 3.

![Graph](image.png)

**Fig. 3.** File name: do1_test2comp.m. Agreement between Fig. 2 and 3 is obvious. Solid lines: AWR, Fig. 1, markers: MATLAB, Eq. (13,14,12)

From Eq. (12) for $A_2 \to 0$:

\[ I_0(A_1)I_1(A_2) = -I_2(A_1)I_0(A_2) \]  

(15)

3. Frequency tripling:

Here $\omega_2 = 3\omega_1$. In the same way, as above, we can derive the circuit equations for DC, $\omega_1$ and $\omega_2$. Eqs. (13-14) remain valid, Eq. (12) will be replaced by the following one:

\begin{align*}
0 &= Z_0 \cdot I_0 \cdot (e^{A_0}[I_0(A_1)]2I_1(A_2) + e^{A_0}[I_0(A_2)]2I_3(A_1)) + A_2
\end{align*}

and accordingly, Eq. (15) is replaced by

\[ I_0(A_1)I_1(A_2) = -I_3(A_1)I_0(A_2) \]  

(17)

The AWR analysis is taken in the circuit of Fig. 1, the analysis results are as follows:
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Fig. 4. Input and output voltage amplitude as functions of the input RF power, for a diode frequency tripler

Fig. 5. Compared to Fig. 4, Matlab analysis results using our equations, for the diode frequency tripler. Filename: tri1_test1comp.m. Agreement between the two sets of results is obvious. Solid lines: AWR, Fig. 1, markers: MATLAB, Eq. (13,14,16)

4. Third order intermodulation:

Here \( \omega_3 = 2\omega_1 - \omega_2 \). We make the base frequency component equal to the same spectrum line derived from other spectrum lines based on Eq. (7) above. Here we take into account DC, \( \omega_1 \), \( \omega_2 \); \( \omega_3 = 2\omega_1 - \omega_2 \) frequency components.

\[
iD = 10\left(e^{A_0+A_1 \cos(\omega_1 t-\varphi_1)+A_2 \cos(\omega_2 t-\varphi_2)+A_3 \cos(\omega_3 t-\varphi_3)+\cdots} - 1\right) =
\]
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\[ I_0(e^{A_0}[I_0(A_1 \cos(\phi_1)) + 2I_1(A_1 \cos(\phi_1)) \cos \omega_1 t + 2I_2(A_1 \cos(2\phi_1)) \cos(2\omega_1)t + ...] \times
\]
\[ * [I_0(A_1 \sin(\phi_1)) + 2I_1(A_1 \sin(\phi_1)) \sin \omega_1 t + 2I_2(A_1 \sin(2\phi_1)) \sin(2\omega_1)t + ...] * 
\]
\[ * [I_0(A_2 \cos(\phi_2)) + 2I_1(A_2 \cos(\phi_2)) \cos \omega_2 t + ...] * 
\]
\[ * [I_0(A_2 \sin(\phi_2)) + 2I_1(A_2 \sin(\phi_2)) \sin \omega_2 t + ...] * 
\]
\[ * [I_0(A_3 \cos(\phi_3)) + 2I_1(A_3 \cos(\phi_3)) \cos \omega_3 t + ...] * 
\]
\[ * [I_0(A_3 \sin(\phi_3)) + 2I_1(A_3 \sin(\phi_3)) \sin \omega_3 t + ...] * ... - 1 \]

(18)

where all \( A_i \)-s are normalized to \( nVT \). The circuit is resistive, all phases are zero.

\[
iD = 
\]
\[ = I_0(e^{A_0}[I_0(A_1) + 2I_1(A_1) \cos \omega_1 t + 2I_2(A_1) \cos(2\omega_1)t + ...] \times 
\]
\[ * [I_0(A_2) + 2I_1(A_2) \cos \omega_2 t + ...] * 
\]
\[ * [I_0(A_3) + 2I_1(A_3) \cos \omega_3 t + ...] - 1 \]

(19)

We substitute \( iD \) into the following circuit equation:

\[
Vg1 * \cos \omega_1 t + Vg2 * \cos \omega_2 t = Z0 * iD + uD
\]

(20)

In the following, we put down the different frequency components of Eq. (20) using Eq. (19).

DC:

\[
0 = Z0 * I_0(e^{A_0}I_0(A_1)I_0(A_2)I_0(A_3) - 1) + A_0
\]

(21)

\[ \omega_1: \]

\[
Vg1 = Z0 * I_0e^{A_0}I_0(A_2)I_0(A_3)2I_1(A_1) + A_1
\]

(22)

\[ \omega_2: \]

\[
Vg2 = Z0 * I_0e^{A_0}I_0(A_1)I_0(A_3)2I_1(A_2) + A_2
\]

(23)

\[ \omega_3 = 2\omega_1 - \omega_2: \]

\[
0 = Z0 * I_0e^{A_0}[I_1(A_2)I_0(A_3)2I_2(A_1) + I_0(A_1)I_0(A_2)2I_1(A_3)] + A_3
\]

(24)

Now we substitute \( Vg = Vg1 = Vg2 \) and \( A_1 = A_2: \)

DC:

\[
0 = Z0 * I_0(e^{A_0}[I_0(A_1)]^2I_0(A_3) - 1) + A_0
\]

(25)

\[ \omega_1: \]

\[
Vg = Z0 * I_0e^{A_0}I_0(A_1)I_0(A_3)2I_1(A_1) + A_1
\]

(26)

\[ \omega_3 = 2\omega_1 - \omega_2: \]
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\[ 0 = Z_0 \ast I_0 e^{A_0} \{ I_1(A_1)I_0(A_3)2I_2(A_1) + [I_0(A_1)]^22I_1(A_3) \} + A_3 \]  

(27)

In Eq. (25-26), \( I_0(A_3) \approx 1 \):

**DC:**

\[ 0 = Z_0 \ast I_0 \{ e^{A_0}[I_0(A_1)]^2-1 \} + A_0 \]  

(28)

**\( \omega_1 \):**

\[ V_g = Z_0 \ast I_0 e^{A_0}I_0(A_1)2I_1(A_1) + A_1 \]  

(29)

We solve Eq. (28-29) for \( A_0 \) and \( A_1 \), then substitute into Eq. (27) for \( A_3 \).

From Eq. (24) for \( A_3 \rightarrow 0 \):

\[ I_2(A_1)I_1(A_2)I_0(A_3) = -I_0(A_1)I_0(A_2)I_1(A_3) \]  

(30)

We analyzed the same circuit in Fig. 1. Circuit analysis results are the following.

**Fig. 6. First and third order intermodulation of the one diode circuit in Fig. 1. by AWR**

**Fig. 7. Intermodulation products of a one diode circuit in Fig. 1 described by our equations (27-29). Agreement with the circuit analysis results in Fig. 6 is obvious. Filename: int2_test1.m. Solid lines: AWR,**
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In the Appendix, from Eq. (30) we show that $V_{out} = V_3$ and $V_{in} = V_1 = V_2$ as functions of input power are parallel lines, as functions of $P_{RF}$ in dBm.

Appendix: From Eq. (30) we show that $V_{out} = V_3$ and $V_{in} = V_1 = V_2$ as functions of input power are parallel lines.

$$\frac{I_2(V_{in})I_1(V_{in})}{I_0(V_{in})I_0(V_{in})} = -\frac{I_1(V_{out})}{I_0(V_{out})}$$  \hspace{1cm} (A1)

We show here the different frequency components normalized to the corresponding DC values. First we redraw Fig. 7 in a broader PRF range.

**Fig. A1. In the PRF range from 0 to +5 dBm, the two intermodulation products are in parallel, a very good approximation**
Looking at Eq. (A1), if the Bessel functions of different orders are in parallel, then the statement is proven. For this purpose, let us plot $I_0$, $I_1$ and $I_2$ in this range, Fig. A2.

\[
I_n(z) = \left(\frac{1}{2} z\right)^n \sum_{k=0}^{\infty} \frac{(\frac{1}{2} z)^k}{k!(n+k)!}
\]  
\[\text{(A2)}\]

We start with Eq. (A1) for small $V_{in}$ and $V_{out}$:

\[
I_2(V_{in})I_1(V_{in}) = -I_1(V_{out})
\]  
\[\text{(A3)}\]

For small arguments:

\[
I_1(V_{in}) = \frac{1}{2} V_{in}
\]  
\[\text{(A4)}\]

\[
I_2(V_{in}) = \left(\frac{1}{2} V_{in}\right)^2 \times \frac{1}{2}
\]  
\[\text{(A5)}\]

\[
I_1(V_{out}) = \frac{1}{2} V_{out}
\]  
\[\text{(A6)}\]

Substituting (A4-A6) into (A3):

\[
\frac{1}{16} V_{in}^3 = -\frac{1}{2} V_{out}
\]  
\[\text{(A7)}\]

De-normalizing the voltages:

\[
V_{out} = \frac{-\frac{1}{2} V_{in}^3}{\frac{8}{nV^2}}
\]  
\[\text{(A8)}\]

that exactly reflects the 1:3 slope relation between first and third order products in dBV scale.

**References:**

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   ieeexplore.ieee.org/document/1444736/

